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A DECISION PROCEDURE FOR A SUBLANGUAGE OF SET THEORY INVOLVING MONOTONE ADDITIVE AND MULTIPLICATIVE FUNCTIONS, II. THE MULTI-LEVEL CASE

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MLSS is a decidable sublanguage of set theory involving the predicates membership, set equality, set inclusion, and the operators union, intersection, set difference, and singleton.

In this paper we extend **MLSS** with constructs for expressing monotonicity, additivity, and multiplicativity properties of set-to-set functions. We prove that the resulting language is decidable by reducing the problem of determining the satisfiability of its sentences to the problem of determining the satisfiability of sentences of **MLSS**.

In addition, we show an interesting model theoretic property of **MLSS**, the *singleton model property*, upon which our decidability proof is based. Intuitively, the singleton model property states that if a formula is satisfiable, then it is satisfiable in a model whose non-empty Venn regions are singleton sets.

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1. Introduction.

Since many mathematical facts can be expressed in set-theoretic terms, it is useful to design and implement a proof system based on the powerful formalism of set theory. However, the expressive power of the full language of set theory comes at the price of undecidability. It is therefore more practical to concentrate on sublanguages of set theory.

Computable set theory [3], [6] is that area of mathematics and computer science which studies the decidability properties of sublanguages of set theory. It was initiated by the seminal paper of [10], which proved the decidability of:

– a *multi-level syllogistic* (**MLS**) involving membership, set equality, set inclusion, union, intersection, and set difference;
– a *multi-level syllogistic with singleton* (**MLSS**) extending **MLS** with the singleton operator.

In this paper we introduce the fragment of set theory **MLSSmf** (*multi-level syllogistic with monotone functions*), which extends **MLSS** with free set-to-set function symbols and the following predicates for expressing monotonicity, additivity, and multiplicativity properties of set-to-set functions:

– $inc(f)$, which holds iff f is *increasing*, that is, $a \subseteq b \rightarrow f(a) \subseteq f(b)$, for all sets a, b;
- $dec(f)$, which holds iff f is *decreasing*, that is, $a \subseteq b \rightarrow f(b) \subseteq f(a)$, for all sets a, b;
– $add(f)$, which holds iff f is *additive*, that is, $f(a \cup b) = f(a) \cup f(b)$, for all sets a, b;
– $mul(f)$, which holds iff f is *multiplicative*, that is, $f(a \cap b) = f(a) \cap f(b)$, for all sets a, b;
– $f \preceq g$, which holds iff $f(a) \subseteq g(a)$, for every set a.

We prove that **MLSSmf** is decidable by providing a reduction algorithm which maps each sentence of **MLSSmf** into an equisatisfiable sentence of **MLSS**. Then the decidability of **MLSSmf** will follow from the decidability of **MLSS**.

An interesting and important model theoretic property of **MLSS**, which we call the *singleton model property*, turned out to be essential for proving the decidability of **MLSSmf**, and gave us also a major insight on the reasons why multilevel syllogistics are decidable. Intuitively, the singleton model property of **MLSS** states that if a suitably normalized formula in the language **MLSS** is satisfiable, then it is satisfiable in a model whose non-empty Venn regions are singleton sets.

1.1. Related work.

Our reduction algorithm is an *augmentation* method, that is, a method that uses as a black box a decision procedure for a language L in order to obtain a decision procedure for a nontrivial extension L' of L. Other augmentation methods for set-theoretic languages can be found in [13], [14].

The literature abounds with decidability results for extensions of **MLSS** involving uninterpreted function symbols. Ferro, Omodeo, and Schwartz [11] and Beckert and Hartmer [1] proved the decidability of an extension of **MLSS** with uninterpreted function symbols, but with no monotonicity, additivity, and multiplicativity constructs. Cantone and Zarba [9] proved the decidability of a sublanguage of set theory with urelements[4] and stratified sets involving monotonicity constructs, but no additivity and multiplicativity constructs. Zarba, Cantone, and Schwartz [15] extended the result of [9] to also include additivity and multiplicativity constructs.

Preliminary versions of the results presented in this paper can be found in [7], [8], and [4].

1.2. Organization of the paper.

The paper is organized as follows. In Section 2 we formally define the syntax and semantics of the languages **MLS**, **MLSS** and **MLSSmf**, and we give other useful notions which will be needed subsequently. In Section 3 we present our reduction algorithm for mapping sentences of **MLSSmf** into equisatisfiable sentences of **MLSS**. In Section 4 we prove that our reduction algorithm is correct and we assess its complexity. In Section 5 we discuss the singleton model property of **MLSS**. Finally, in Section 6 we draw conclusions from our work.

2. Preliminaries.

2.1. Von Neumann hierarchy of sets.

Most of the satisfiability problems studied in Computable Set Theory, included the one in the present paper, are relative to the von Neumann hierarchy

[4] Urelements (also known as atoms or individuals) are objects which contain no elements but are distinct from the empty set. "Ur" is a German prefix meaning "primitive" or "original".

of sets $\mathcal{V}$ defined inductively by:

$$\begin{array}{lll} \mathcal{V}_0 & = \emptyset, & \\ \mathcal{V}_{\alpha+1} & = \mathcal{P}(\mathcal{V}_\alpha), & \text{for each ordinal } \alpha, \\ \mathcal{V}_\lambda & = \bigcup_{\mu<\lambda} \mathcal{V}_\mu, & \text{for each limit ordinal } \lambda, \\ \mathcal{V} & = \bigcup_{\alpha\in\mathcal{O}} \mathcal{V}_\alpha, & \end{array}$$

where $\mathcal{P}$ denotes the power-set operator and $\mathcal{O}$ is the class of all ordinals. The axiom of regularity implies that every set belongs to some set $\mathcal{V}_\alpha$ (see for example [12]). Therefore for every set a we may define its *rank* by putting

$$rank(a) = \text{least } \alpha \text{ such that } a \in \mathcal{V}_{\alpha+1}.$$

We mention two very useful properties of the rank function which will be used later:

(i) if $a \in b$, then $rank(a) < rank(b)$;
(ii) if $a \subseteq b$, then $rank(a) \leq rank(b)$.

2.2. Multi-level syllogistic.

MLS (*multi-level syllogistic*) is the unquantified set-theoretic language containing:

– an infinitely enumerable collection of variables;
– the constant $\emptyset$ (*empty set*);
– the operators $\cup$ (*union*), $\cap$ (*intersection*), and $\setminus$ (*set difference*);
– the predicates $\in$ (*membership*), $=$ (*set equality*), and $\subseteq$ (*set inclusion*);
– the propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

Example 1. Let x, y, z be variables. Then the expression

$$\neg[(x \setminus y = \emptyset \wedge z \in x) \rightarrow y \neq \emptyset] \tag{1}$$

is an example of an **MLS**-formula.

An *assignment* $\mathcal{A}$ over a collection of variables V is any map from V into the von Neumann hierarchy of sets $\mathcal{V}$. Given an **MLS**-formula φ over a collection V of variables, and an assignment $\mathcal{A}$ over V, we denote with $\varphi^{\mathcal{A}}$ the truth-value of φ obtained by interpreting each variable $x \in V$ with the set $x^{\mathcal{A}}$, and interpreting the set symbols and logical connectives according to their standard meaning. A *model* of an **MLS**-formula φ is an assignment $\mathcal{A}$ such that $\varphi^{\mathcal{A}}$ is true. An **MLS**-formula φ is *satisfiable* if it has a model.

The satisfiability problem for **MLS** is the problem of determining whether or not an **MLS**-formula φ is satisfiable. This problem is decidable [10].

Example 2. The **MLS**-formula(1) in Example 1 is not satisfiable. In fact, for every assignment $\mathcal{A}$, if $x^{\mathcal{A}} \setminus y^{\mathcal{A}} = \emptyset$ and $z^{\mathcal{A}} \in x^{\mathcal{A}}$, it must be the case that $z^{\mathcal{A}} \in x^{\mathcal{A}} \cap y^{\mathcal{A}}$, which implies that $y^{\mathcal{A}} \neq \emptyset$.

2.3. Extensions of MLS.

MLSS (*multi-level syllogistic with singleton*) is the unquantified set-theoretic language extending **MLS** with the singleton operator $\{\cdot\}$. The semantics of **MLSS** is defined similarly to the semantics of **MLS**. The satisfiability problem for **MLSS** is decidable [10].

Example 3. Let x, y, z be variables. Then the expression

$$\neg[(x = \{y\} \wedge x = y \cup z) \rightarrow y = \emptyset]$$

is an example of an unsatisfiable **MLSS**-formula. In fact, $x = \{y\} \wedge x = y \cup z \wedge y \neq \emptyset$ implies $y \in y$, which contradicts the regularity axiom.

In this paper we focus on the satisfiability problem for the unquantified set-theoretic language **MLSSmf** (*multi-level syllogistic with singleton and monotone functions*), which extends **MLSS** with an infinitely enumerable collection of unary set-to-set function symbols and the predicates inc, dec, add, mul, and $\preceq$.

The semantics of **MLSSmf** is defined similarly to the semantics of **MLS** and **MLSS**, with the only difference that if $\mathcal{A}$ is an (**MLSSmf**-)assignment and f is a function symbol then $f^{\mathcal{A}}$ is a class function from $\mathcal{V}$ into $\mathcal{V}$. Moreover, for any assignment $\mathcal{A}$ we agree that:

- $inc(f)$ holds in $\mathcal{A}$ if and only if $f^{\mathcal{A}}$ is *increasing*, that is, $s \subseteq t \rightarrow f^{\mathcal{A}}(s) \subseteq f^{\mathcal{A}}(t)$, for all sets s, t;
- $dec(f)$ holds in $\mathcal{A}$ if and only if $f^{\mathcal{A}}$ is *decreasing*, that is, $s \subseteq t \rightarrow f^{\mathcal{A}}(t) \subseteq f^{\mathcal{A}}(s)$, for all sets s, t;
- $add(f)$ holds in $\mathcal{A}$ if and only if $f^{\mathcal{A}}$ is *additive*, that is, $f^{\mathcal{A}}(s \cup t) = f^{\mathcal{A}}(s) \cup f^{\mathcal{A}}(t)$, for all sets s, t;
- $mul(f)$ holds in $\mathcal{A}$ if and only if $f^{\mathcal{A}}$ is *multiplicative*, that is, $f^{\mathcal{A}}(s \cap t) = f^{\mathcal{A}}(s) \cap f^{\mathcal{A}}(t)$, for all sets s, t;
- $f \preceq g$ holds in $\mathcal{A}$ if and only if $f^{\mathcal{A}}(s) \subseteq g^{\mathcal{A}}(s)$, for every set s.

Example 4. Let f, g be two set-to-set function symbols, and let x be a variable. Then the **MLSSmf**-formula

$$\neg[(add(f) \wedge f \preceq g) \rightarrow f(\emptyset) \subseteq g(\{x\})]$$

is unsatisfiable. Intuitively, this is due to the fact that every additive function is also increasing.

Example 5. Let $f, g : \mathcal{V} \to \mathcal{V}$ be two class functions, and denote with $f \circ g$, $f \cup g$, and $f \cap g$ the class functions defined by:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)),\\ (f \cup g)(x) &= f(x) \cup g(x),\\ (f \cap g)(x) &= f(x) \cap g(x).\end{aligned}$$

Then, for any two additive class functions f_1, f_2, both $f_1 \circ f_2$ and $f_1 \cup f_2$ are additive. Moreover, for any two multiplicative class functions g_1, g_2, both $g_1 \circ g_2$ and $g_1 \cap g_2$ are multiplicative.

Example 6. Let $G : \mathcal{V} \to \{\mathbf{false}, \mathbf{true}\}$ be a class predicate, and let f_G be the class function defined by

$$f_G(x) = \{y \in x : G(y)\}.$$

Then f_G is both additive and multiplicative.

Example 7. The unary union function Un, defined by $Un(x) = \bigcup_{y \in x} y$, is additive. The power-set function $\mathcal{P}$, defined by $\mathcal{P}(x) = \{y : y \subseteq x\}$, is multiplicative. The cartesian product function $\times$, defined by $x \times y = \{(u, v) : u \in x \text{ and } v \in y\}$, is additive and multiplicative in both arguments.

2.4. Normalized literals.

In order to simplify details, we will often consider conjunctions of *normalized* **MLSSmf**-literals of the form:

$$\begin{array}{llll} x = y, & x \neq y, & x = y \cup z, & x = y \setminus z, \\ x = \{y\}, & x = f(y), & inc(f), & dec(f), \\ add(f), & mul(f), & f \preceq g. & \end{array} \tag{2}$$

Let φ be an **MLSSmf**-formula. By suitably introducing new variables, it is possible to convert φ into an equisatisfiable formula $\psi = \psi_1 \vee \cdots \vee \psi_k$ in disjunctive normal form, where each ψ_i is a conjunction of normalized **MLSSmf**-literals of the form (2). Thus, we have the following result.

Lemma 8. *The satisfiability problem for* **MLSSmf**-*formulae is equivalent to the satisfiability problem for conjunctions of normalized* **MLSSmf**-*literals of the form (2).*

Similar results to the one in Lemma 8 also hold for the languages **MLS** and **MLSS**, although with different groups of normalized literals.

Lemma 9. *The satisfiability problem for* **MLS**-*formulae is equivalent to the satisfiability problem for conjunctions of normalized* **MLS**-*literals of the form:*

$$x = y, \quad x \neq y, \quad x = y \cup z, \quad x = y \setminus z, \quad x \in y. \tag{3}$$

Lemma 10. *The satisfiability problem for* **MLSS**-*formulae is equivalent to the satisfiability problem for conjunctions of normalized* **MLSS**-*literals of the form:*

$$x = y, \quad x \neq y, \quad x = y \cup z, \quad x = y \setminus z, \quad x = \{y\}. \tag{4}$$

Unless otherwise specified, in the rest of this paper the word normalized refers to literals of the form (2).

3. The reduction algorithm.

Let $\mathcal{C}$ be a conjunction of normalized **MLSSmf**-literals, and denote with $V = \{x_1, \ldots, x_n\}$ and F the collections of variables and function symbols occurring in $\mathcal{C}$, respectively. In this section we describe a reduction algorithm for converting $\mathcal{C}$ into an equisatisfiable conjunction $\mathcal{C}^*$ of **MLSS**-formulae.

We will use the following notation. Given a set a, $\mathcal{P}^+(a)$ denotes the set $\mathcal{P}(a) \setminus \{\emptyset\}$. Moreover, we denote with ℓ_j the set $\{\alpha \in \mathcal{P}^+(\{1, \ldots, n\}) : j \in \alpha\}$, for $1 \leq j \leq n$.

The reduction algorithm is shown in Figure 1, and consists of three steps.

In the first step, we generate new variaboes whose intuitive meaning is as follows:

- for each $\alpha \in \mathcal{P}^+(\{1, \ldots, n\})$, the new variable v_α is intented to represent the Venn region $\bigcap_{i \in \alpha} x_i \setminus \bigcup_{j \notin \alpha} x_j$;
- for each $\ell \subseteq \mathcal{P}^+(\{1, \ldots, n\})$, the new variable $w_{f,\ell}$ is intended to represent the image of the set $\bigcup_{\alpha \in \ell} v_\alpha$ under the function f.

In the second step, we add to $\mathcal{C}$ appropriate **MLSS**-formulae whose purpose is to model the variables v_α and $w_{f,\ell}$ according to their intuitive

Reduction algorithm

Input: a conjunction $\mathcal{C}$ of normalized **MLSSmf**-literals
Output: a conjunction $\mathcal{C}^*$ of **MLSSmf**-formulae

Notation

- $V = \{x_1, \ldots, x_n\}$ is the collection of variables occurring in $\mathcal{C}$;
- F is the collection of function symbols occurring in $\mathcal{C}$;
- $\mathcal{P}^+(a) = \mathcal{P}(a) \setminus \{\emptyset\}$, for each set a;
- ℓ_j stands for the set $\{\alpha \in \mathcal{P}^+(\{1, \ldots, n\}) : j \in \alpha\}$, for $1 \leq j \leq n$.

Step 1. Generate the following new variables:

$$v_\alpha, \qquad \text{for each } \alpha \in \mathcal{P}^+(\{1, \ldots, n\}),$$
$$w_{f,\ell}, \qquad \text{for each } f \in F \text{ and } \ell \subseteq \mathcal{P}^+(\{1, \ldots, n\}).$$

Step 2. Add to $\mathcal{C}$ the following **MLSS**-formulae:

$$v_\alpha = \bigcup_{i \in \alpha} x_i \setminus \bigcap_{j \notin \alpha} x_j, \qquad \text{for each } \alpha \in \mathcal{P}^+(\{1, \ldots, n\}),$$

and

$$\bigcup_{\alpha \in \ell} v_\alpha = \bigcup_{\beta \in m} v_\beta \rightarrow w_{f,\ell} = w_{f,m} \qquad \text{for each } f \in F \text{ and } \ell, m \subseteq \mathcal{P}^+(\{1, \ldots, n\}).$$

Step 3. Replace literals in $\mathcal{C}$ containing function symbols with **MLSS**-formulae as follows:

$$x_i = f(x_j) \quad \Longrightarrow \quad x_i = w_{f,l_j}$$
$$inc(f) \quad \Longrightarrow \quad \bigwedge_{\ell \subseteq m} (w_{f,\ell} \subseteq w_{j,m}$$
$$dec(f) \quad \Longrightarrow \quad \bigwedge_{\ell \subseteq m} (w_{f,m} \subseteq w_{j,\ell}$$
$$add(f) \quad \Longrightarrow \quad \bigwedge_{\ell,m} (w_{f,\ell \cup m} = w_{f,\ell} \cup w_{f,m}$$
$$mul(f) \quad \Longrightarrow \quad \bigwedge_{\ell,m} (w_{f,\ell \cap m} = w_{f,\ell} \cap w_{f,m}$$
$$f \preceq g \quad \Longrightarrow \quad \bigwedge_{\ell} (w_{f,\ell} \subseteq w_{g,\ell}$$

Figure 1: The reduction algorithm.

meaning. In particular, the variables $w_{f,\ell}$ are modeled by noticing that for each $\ell, m \subseteq \mathcal{P}^+(\{1, \ldots, n\})$, if $\bigcup_{\alpha \in \ell} v_\alpha = \bigcup_{\beta \in m} v_\beta$ then $f\big(\bigcup_{\alpha \in \ell} v_\alpha\big) = f\Big(\bigcup_{\beta \in m} v_\beta\Big)$.

Finally, in the third step we remove from $\mathcal{C}$ all literals involving function symbols. This is done by replacing all literals of the form $x_i = f(x_j)$, $inc(f)$, $dec(f)$, $add(f)$, $mul(f)$, and $f \preceq g$ with **MLSS**-literals involving only the variables x_i and the new variables $w_{f,\ell}$.

We claim that our reduction algorithm is correct. More specifically, we claim that if $\mathcal{C}^*$ is the result of applying to $\mathcal{C}$ our reduction algorithm then:

- the reduction is *sound*, namely, if $\mathcal{C}$ is satisfiable, so is $\mathcal{C}^*$;
- the reduction is *complete*, namely, if $\mathcal{C}^*$ is satisfiable, so is $\mathcal{C}$.

The next section proves that our reduction algorithm is sound and complete, and therefore it yields a decision procedure for **MLSSmf**.

4. Correctness.

4.1. Soundness.

Let $\mathcal{C}$ be a satisfiable conjunction of normalized **MLSSmf**-literals, and let $\mathcal{C}^*$ be the result of applying to $\mathcal{C}$ the reduction algorithm in Figure 1. The key idea of the soundness proof is that, given a model $\mathcal{A}$ of $\mathcal{C}$, a model $\mathcal{B}$ of $\mathcal{C}^*$ can be constructed in the most natural way if we remember the intuitive meaning of the variables v_α and $w_{f,\ell}$.

Lemma 11. (Soundness). *Let $\mathcal{C}$ be a conjunction of normalized* **MLSSmf**-*literals, and let $\mathcal{C}^*$ be the result of applying to $\mathcal{C}$ the reduction algorithm in Figure 1. Then if $\mathcal{C}$ is satisfiable, so is $\mathcal{C}^*$.*

Proof. Let $\mathcal{A}$ be a model of $\mathcal{C}$, and denote with $V = \{x_1, \ldots, x_n\}$ and F the collections of variables and function symbols occurring in $\mathcal{C}$, respectively. We claim that the assignment $\mathcal{B}$ defined by:

$$
\begin{array}{ll}
x_i^{\mathcal{B}} = x_i^{\mathcal{A}}, & \text{for each } i \in \{1, \ldots, n\}, \\
v_\alpha^{\mathcal{B}} = \bigcap_{i \in \alpha} x_i^{\mathcal{A}} \setminus \bigcup_{j \notin \alpha} x_j^{\mathcal{A}}, & \text{for each } \alpha \in \mathcal{P}^+(\{1, \ldots, n\}), \\
w_{f,\ell}^{\mathcal{B}} = f^{\mathcal{A}}\Big(\bigcup_{\alpha \in \ell} v_\alpha^{\mathcal{B}}\Big), & \text{for each } f \in F \text{ and } \ell \subseteq \mathcal{P}^+(\{1, \ldots, n\})
\end{array}
$$

is a model of $\mathcal{C}^*$.

To see this, note that formulae in $\mathcal{C}^*$ can be partitioned into the following categories:

(I) literals of the form $x = y$, $x \neq y$, $x = y \cup z$, $x = y \setminus z$ and $x = \{y\}$ originally present in $\mathcal{C}$;

(II) literals of the form $v_\alpha = \bigcup_{i \in \alpha} x_i \setminus \bigcap_{j \notin \alpha} x_j$;

(III) formulae of the form $\bigcup_{\alpha \in \ell} v_\alpha = \bigcup_{\beta \in m} v_\beta \rightarrow w_{f,\ell} = w_{f,m}$;

(IV) literals of the form $x_i = w_{f,\ell_j}$, which replace literals of the form $x_i = f(x_j)$ in $\mathcal{C}$;

(V) conjunctions replacing literals of the form $inc(f)$, $dec(f)$, $add(f)$, $mul(f)$, and $f \preceq g$ in $\mathcal{C}$.

Literals of type (I) are true in $\mathcal{B}$ because they contain only variables in V, and by construction $\mathcal{B}$ agrees with $\mathcal{A}$ on all such variables.

Concerning literals of type (II), we have

$$v_\alpha^{\mathcal{B}} = \bigcup_{i \in \alpha} x_i^{\mathcal{A}} \setminus \bigcap_{j \notin \alpha} x_j^{\mathcal{A}} = \bigcup_{i \in \alpha} x_i^{\mathcal{B}} \setminus \bigcap_{j \notin \alpha} x_j^{\mathcal{B}}.$$

Concerning literals of type (III), let $\bigcup_{\alpha \in \ell} v_\alpha^{\mathcal{B}} = \bigcup_{\beta \in m} v_\beta^{\mathcal{B}}$. Then

$$f^{\mathcal{A}}\Big(\bigcup_{\alpha \in \ell} v_\alpha^{\mathcal{B}}\Big) = f^{\mathcal{A}}\Big(\bigcup_{\beta \in m} v_\beta^{\mathcal{B}}\Big),$$

for every $f \in F$. Thus,

$$w_{f,\ell}^{\mathcal{B}} = f^{\mathcal{A}}\Big(\bigcup_{\alpha \in \ell} v_\alpha^{\mathcal{B}}\Big) = f^{\mathcal{A}}\Big(\bigcup_{\beta \in m} v_\beta^{\mathcal{B}}\Big) = w_{f,m}^{\mathcal{B}},$$

for every $f \in F$. Concerning literals of type (IV), let $x_i = f(x_j)$ be in $\mathcal{C}$. Then we have

$$x_i^{\mathcal{B}} = x_i^{\mathcal{A}} = f^{\mathcal{A}}\Big(x_j^{\mathcal{A}}\Big) = f^{\mathcal{A}}\Big(\bigcup_{\alpha \in \ell_j} v_\alpha^{\mathcal{B}}\Big) = w_{f,\ell_j}^{\mathcal{B}},$$

where $\ell_j = \{\alpha \in \mathcal{P}^+(\{1, \ldots, n\}) : j \in \alpha\}$.

Concerning literals of type (V), consider a literal of the form $inc(f)$ in $\mathcal{C}$ and suppose that $\ell \subseteq m$, with $\ell, m \subseteq \mathcal{P}^+(\{1, \ldots, n\})$. We want to show that the literal $w_{f,\ell} \subseteq w_{f,m}$ occurring in $\mathcal{C}^*$ is true in $\mathcal{B}$. To do so, observe that, since $\ell \subseteq m$, we have $\bigcup_{\alpha \in \ell} v_\alpha^{\mathcal{B}} \subseteq \bigcup_{\beta \in m} v_\beta^{\mathcal{B}}$. Since $f^{\mathcal{A}}$ is increasing, we also have

$f^{\mathcal{A}}\Big(\bigcup_{\alpha\in\ell} v_\alpha^{\mathcal{B}}\Big) \subseteq f^{\mathcal{A}}\Big(\bigcup_{\beta\in m} v_\beta^{\mathcal{B}}\Big)$, and therefore $w_{f,\ell}^{\mathcal{B}} \subseteq w_{f,m}^{\mathcal{B}}$. The cases regarding conjunctions introduced in $\mathcal{C}^*$ to replace literals of the form $dec(f)$, $add(f)$, $mul(f)$ and $add(f)$ can be treated similarly. $\square$

4.2. Completeness.

Let $\mathcal{C}$ be a conjunction of normalized **MLSSmf**-literals. As before, let us denote with $V = \{x_1, \ldots, x_n\}$ and F the collections of variables and function symbols occurring in $\mathcal{C}$, respectively. Also, let $\mathcal{C}^*$ be the result of applying to $\mathcal{C}$ the reduction algorithm in Figure 1. To show the completeness of our reduction algorithm, we need to prove that if $\mathcal{C}^*$ is satisfiable, so is $\mathcal{C}$.

To do so, let $\mathcal{B}$ be a model of $\mathcal{C}^*$, and let us start to define an assignment $\mathcal{M}$ over the variables and function symbols in $\mathcal{C}$ by letting

$$x_i^{\mathcal{M}} = x_i^{\mathcal{B}}, \qquad \text{for each } i = 1, \ldots, n.$$

In order to define $\mathcal{M}$ over the function symbols in F, let us recall that the intuitive meaning of a variable of the form $w_{f,\ell}$ is to represent the expression $f(\bigcup_{\alpha\in\ell} v_\alpha)$. Thus, our definition of $f^{\mathcal{M}}$ should satisfy the property that $f^{\mathcal{M}}(\bigcup_{\alpha\in\ell} v_\alpha^{\mathcal{B}}) = w_{f,\ell}^{\mathcal{B}}$, for every $\ell \subseteq \mathcal{P}^+(\{1, \ldots, n\})$. But how do we define $f^{\mathcal{M}}(a)$ in the more general case in which a is not the union of sets of the form $v_\alpha^{\mathcal{B}}$? The idea is to define suitably a *discretization function* $\lambda : \mathcal{V} \to \mathcal{P}(\mathcal{P}^+(\{1, \ldots, n\}))$, and then let

$$f^{\mathcal{M}}(a) = w_{f,\lambda(a)}^{\mathcal{B}}, \qquad \text{for each } f \in F \text{ and each set } a.$$

To achieve completeness, we need a *good* discretization function.

Definition 12. Let $\mathcal{C}$ be a conjunction of normalized **MLSSmf**-literals and let $\mathcal{C}^*$ be the result of applying to $\mathcal{C}$ the reduction algorithm in Figure 1. A discretization function $\lambda : \mathcal{V} \to \mathcal{P}(\mathcal{P}^+(\{1, \ldots, n\}))$ is **good** with respect to a model $\mathcal{B}$ of $\mathcal{C}^*$ if the following conditions hold:

(A) λ is increasing;
(B) λ is additive;
(C) λ is multiplicative;
(D) if $a = \bigcup_{\alpha\in\ell} v_\alpha^{\mathcal{B}}$ then $a = \bigcup_{\alpha\in\lambda(a)} v_\alpha^{\mathcal{B}}$, for each $\ell \subseteq \mathcal{P}^+(\{1, \ldots, n\})$.

Lemma 13. *Let $\mathcal{C}$ be a conjunction of normalized* **MLSSmf**-*literals, let $\mathcal{C}^*$ be the result of applying to $\mathcal{C}$ the reduction algorithm in Figure 1, and let $\mathcal{B}$ be a model of $\mathcal{C}^*$. Assume that there exists a discretization function $\lambda : \mathcal{V} \to \mathcal{P}(\mathcal{P}^+(\{1, \ldots, n\}))$ which is good with respect to $\mathcal{B}$.*

Then $\mathcal{C}$ is satisfiable.

Proof. Let $V = \{x_1, \ldots, x_n\}$ (resp. F) be the collection of variables (resp. function symbols) occurring in $\mathcal{C}$. We now prove that the assignment $\mathcal{M}$ defined by

$$\begin{aligned} x_i^{\mathcal{M}} &= x_i^{\mathcal{B}}, && \text{for each } i = 1, \ldots, n, \\ f^{\mathcal{M}}(a) &= w_{f,\lambda(a)}^{\mathcal{B}}, && \text{for each } f \in F \text{ and each set } a \in \mathcal{V}, \end{aligned}$$

is a model of $\mathcal{C}$ by showing that $\mathcal{M}$ satisfies all literals occurring in $\mathcal{C}$.

Literals of the form $x = y$, $x \neq y$, $x = y \cup z$, $x = y \setminus z$, $x = \{y\}$. This literals are true in $\mathcal{M}$ since $\mathcal{M}$ agrees with $\mathcal{B}$ on all variables in V.

Literals of the form $x_i = f(x_j)$. Let $a = \bigcup_{\alpha \in \ell_j} v_\alpha^{\mathcal{B}}$, with $\ell_j = \{\alpha \in \mathcal{P}^+(\{1, \ldots, n\}) : j \in \alpha\}$. Then, by property (D) of Definition 12, we have $a = \bigcup_{\alpha \in \lambda(a)} v_\alpha^{\mathcal{B}}$, and therefore $w_{f,\ell_j}^{\mathcal{B}} = w_{f,\lambda(a)}^{\mathcal{B}}$. Since the literal $x_i = w_{f,\ell_j}$ is in $\mathcal{C}^*$, we have $x_i^{\mathcal{M}} = x_i^{\mathcal{B}} = w_{f,\ell_j}^{\mathcal{B}} = w_{f,\lambda(a)}^{\mathcal{B}} = f^{\mathcal{M}}(a) = f^{\mathcal{M}}\Big(\bigcup_{\alpha \in \ell_j} v_\alpha^{\mathcal{B}}\Big) = f^{\mathcal{M}}\big(x_j^{\mathcal{B}}\big) = f^{\mathcal{M}}\big(x_j^{\mathcal{M}}\big)$.

Literals of the form $inc(f)$, $dec(f)$, $add(f)$, $mul(f)$ and $f \preceq g$. Let the literal $inc(f)$ be in $\mathcal{C}$, and let $a \subseteq b$. Since λ is increasing, $\lambda(a) \subseteq \lambda(b)$, so that the literal $w_{f,\lambda(a)} \subseteq w_{f,\lambda(b)}$ must be in $\mathcal{C}^*$. We have $f^{\mathcal{M}}(a) = w_{f,\lambda(a)}^{\mathcal{B}} \subseteq w_{f,\lambda(b)}^{\mathcal{B}} = f^{\mathcal{M}}(b)$, implying that $f^{\mathcal{M}}$ is increasing. Similarly one can prove that also all literals of the form $dec(f)$, $add(f)$, $mul(f)$ and $f \preceq g$ in $\mathcal{C}$ are satisfied by $\mathcal{M}$. $\square$

Remark 14. By a careful analysis of the above proof, it turns out that the monotonicity of the discretization function λ is only needed to show the satisfiability of literals of the form $inc(f)$ and $dec(f)$. Likewise, the additivity of λ is needed only for literals of the form $add(f)$ and the multiplicativity of λ is needed only for literals of the form $mul(f)$. Finally, property (D) of λ (cf. Definition 12) is only needed to prove the satisfiability of literals of the form $x_i = f(x_j)$.

Lemma 13 shows that the existence of good discretization functions is enough to ensure the completeness of our reduction algorithm. But how do we define good discretization functions?

As a first attempt, given an arbitrary model $\mathcal{B}$ of $\mathcal{C}^*$, let us put

$$\lambda^+_{\mathcal{B}}(a) = \{\alpha \in \mathcal{P}^+(\{1, \ldots, n\}) : v^{\mathcal{B}}_\alpha \cap a \neq \emptyset\}, \qquad \text{for each set } a.$$

It is easy to see that $\lambda^+_{\mathcal{B}}$ satisfies properties (A), (B), and (D) of Definition 12. However, in general $\lambda^+_{\mathcal{B}}$ is not multiplicative. As a counter-example, assume that there exist two disjoint sets a, b and some $\alpha \subseteq \mathcal{P}^+(\{1, \ldots, n\})$ such that $a \cap v^{\mathcal{B}}_\alpha \neq \emptyset$ and $b \cap v^{\mathcal{B}}_\alpha \neq \emptyset$. Then $\alpha \in \lambda^+_{\mathcal{B}}(a) \cap \lambda^+_{\mathcal{B}}(b)$ but $\lambda^+_{\mathcal{B}}(a \cap b) = \emptyset$.

By Remark 14, in the proof of Lemma 13 the hypothesis that $\lambda^+_{\mathcal{B}}$ is multiplicative is used only to show that the literals of the form $mul(f)$ in $\mathcal{C}$ are satisfied by $\mathcal{M}$. Therefore, if we define **MLSSmf**$^+$ to be the language obtained from **MLSSmf** by removing the symbol mul, we get the following partial result.

Lemma 15. *Let $\mathcal{C}$ be a conjunction of normalized* **MLSSmf**$^+$ *-literals and let $\mathcal{C}^*$ be the result of applying to $\mathcal{C}$ the reduction algorithm in Figure 1. Then if $\mathcal{C}^*$ is satisfiable, so is $\mathcal{C}$.*

By combining Lemma 11 and Lemma 15 we obtain immediately the decidability of **MLSSmf**$^+$.

Theorem 16. *The satisfiability problem for* **MLSSmf**$^+$ *is decidable.*

As a second attempt to find a good discretization function, let us put

$$\lambda^\times_{\mathcal{B}}(a) = \{\alpha \in \mathcal{P}^+(\{1, \ldots, n\} : \emptyset \neq v^{\mathcal{B}}_\alpha \subseteq a\}, \qquad \text{for each set } a.$$

It is easy to see that $\lambda^\times_{\mathcal{B}}$ satisfies properties (A), (C), and (D) of Definition 12. However, in general $\lambda^\times_{\mathcal{B}}$ is not additive. As a counter-example, assume that there exist two sets a, b and some $\alpha \subseteq \mathcal{P}^+(\{1, \ldots, n\})$ such that $v^{\mathcal{B}}_\alpha \subseteq a \cup b$, $v^{\mathcal{B}}_\alpha \not\subseteq a$, and $v^{\mathcal{B}}_\alpha \not\subseteq b$. Then $\alpha \in \lambda^\times_{\mathcal{B}}(a \cup b)$ but $\alpha \notin \lambda^\times_{\mathcal{B}}(a) \cup \lambda^\times_{\mathcal{B}}(b)$.

By Remark 14, in the proof of Lemma 13 the hypothesis that $\lambda^\times_{\mathcal{B}}$ is additive is used only to show that the literals of the form $add(f)$ in $\mathcal{C}$ are satisfied by $\mathcal{M}$. Therefore, if we define **MLSSmf**$^\times$ to be the language obtained from **MLSSmf** by removing the symbol add, we get the following partial result.

Lemma 17. *Let $\mathcal{C}$ be a conjunction of normalized* **MLSSmf**$^\times$ *-literals and let $\mathcal{C}^*$ be the result of applying to $\mathcal{C}$ the reduction algorithm in Figure 1. Then if $\mathcal{C}^*$ is satisfiable, so is $\mathcal{C}$.*

By combining Lemma 11 and Lemma 17 we obtain at once the decidability of **MLSSmf**$^\times$.

Theorem 18. *The satisfiability problem for* **MLSSmf**$^\times$ *is decidable.*

So far, it appears as neither $\lambda^{+}_{\mathcal{B}}$ nor $\lambda^{\times}_{\mathcal{B}}$ are good discretization functions. However, assume that we have a model $\mathcal{B}$ of $\mathcal{C}^*$ such that:

$$(5) \qquad |v_\alpha^{\mathcal{B}}| \leq 1, \qquad \text{for each } \alpha \in \mathcal{P}^+(\{1, \ldots, n\}).$$

Then it is easy to see that in this case $\lambda^{+}_{\mathcal{B}}$ and $\lambda^{\times}_{\mathcal{B}}$ coincide, and therefore they are both additive and multiplicative. Thus, both $\lambda^{+}_{\mathcal{B}}$ and $\lambda^{\times}_{\mathcal{B}}$ are good discretization functions with respect to any model $\mathcal{B}$ of $\mathcal{C}^*$ satisfying (5).

But do models of $\mathcal{C}^*$ satisfying (5) exist? The following lemma gives an affirmative answer to this question.

Lemma 19. *Let $\mathcal{C}$ be a conjunction of normalized* **MLSSmf**-*literals, and let $\mathcal{C}^*$ be the result of applying to $\mathcal{C}$ the reduction algorithm in Figure 1. Assume also that $\mathcal{C}^*$ is satisfiable. Then there exists a model $\mathcal{B}$ of $\mathcal{C}^*$ such that $|v_\alpha^{\mathcal{B}}| \leq 1$, for each $\alpha \in \mathcal{P}^+(\{1, \ldots, n\})$.*

The proof of Lemma 19 is very technical and will be given in Section 5. For now, we just mention that the result of Lemma 19 is a consequence of a model theoretic property of conjunctions of normalized **MLSS**-literals, called the *singleton model property*, which will be discussed in more detail in Section 5.

Combining Lemma 19 with Lemma 13 we can finally obtain the completeness of our reduction algorithm, using either $\lambda^{+}_{\mathcal{B}}$ or $\lambda^{\times}_{\mathcal{B}}$ as a good discretization function with respect to a model $\mathcal{B}$ of $\mathcal{C}^*$ satisfying (5).

Lemma 20. (Completeness). *Let $\mathcal{C}$ be a conjunction of normalized* **MLSSmf**-*literals and let $\mathcal{C}^*$ be the result of applying to $\mathcal{C}$ the reduction algorithm in Figure 1. Then if $\mathcal{C}^*$ is satisfiable, so is $\mathcal{C}$.*

Combining Lemma 11 and Lemma 20, we obtain the decidability of **MLSSmf**.

Theorem 21. (Decidability). *The satisfiability problem for* **MLSSmf** *is decidable.*

4.3. Complexity issues.

Let $\mathcal{C}$ be a conjunction of normalized **MLSSmf**-literals containing n distinct variables and m distinct function symbols. It turns easily out that the formula $\mathcal{C}^*$, which results by applying to $\mathcal{C}$ the reduction algorithm in Figure 1, involves $\mathcal{O}(2^n)$ variables of type v_α, with $\alpha \in \mathcal{P}^+(\{1, \ldots, n\})$, and $\mathcal{O}(m \cdot 2^{2^n})$ variables of type $w_{f,\ell}$, where f is a function symbol in $\mathcal{C}$ and

$\ell \subseteq \mathcal{P}^+(\{1, \ldots, n\})$. Moreover, the collective size of all formulae generated in Step 2 is

$$\mathcal{O}(n \cdot 2^n) + \mathcal{O}(m \cdot 2^n \cdot 2^{2^{n+1}}),$$

and the collective size of all formulae generated in Step 3 is bounded by

$$\mathcal{O}(p \cdot 2^{2^{n+1}}),$$

where p is the number of literals in $\mathcal{C}$ of type

$$x = f(y), \quad inc(f), \quad dec(f), \quad add(f), \quad mul(f), \quad f \preceq g.$$

Thus, if we denote with K the size of $\mathcal{C}$, since $m, n, p \leq K$, we have the following upper bound on the size of $\mathcal{C}^*$:

$$\mathcal{O}(K \cdot 2^K \cdot 2^{2^{K+1}}).$$

Finally, to estimate the complexity of our decision procedure, we must take into account that the formula $\mathcal{C}^*$ must then be tested for satisfiability, and it is known that the satisfiability problem for **MLSS** is NP-complete [5]. Though the satisfiability test for **MLSS** is quite efficient in practice, it becomes very expensive when run on such large formulae as $\mathcal{C}^*$.

5. The singleton model property.

The language **MLSS** enjoys an interesting model-theoretic property: the *singleton model property*. This property states that if $\mathcal{C}$ is a satisfiable conjunction of normalized **MLSS**-literals of the form (4), then there exists a model $\mathcal{M}$ of $\mathcal{C}$ whose non-empty Venn regions all are singleton sets.

We shall derive the singleton model property for conjunctions of normalized **MLSS**-literals as a by-product of a proof of the decidability of **MLSS**. The proof which will be given below is based on the *places* approach and follows much the same lines of [2].

5.1. Places and Venn regions.

The notion of place has been one of the most important and successful tools in computable set theory for deriving decidability results for several extensions of **MLS**. Intuitively, places are a syntactic device for representing the non-empty proper regions of the Venn diagram of a given model.

Definition 22. (Venn diagrams and regions). A VENN DIAGRAM is any finite collection of sets.

Let $D = \{a_1, \ldots, a_n\}$ be a Venn diagram. A set $a \in \mathcal{V}$ is a PROPER VENN REGION of D if $a = \bigcap_{i\in\alpha} a_i \setminus \bigcup_{j\notin\alpha} a_j$, for some non-empty $\alpha \subseteq \{1, \ldots, n\}$. The NON-PROPER VENN REGION of D is the class $\mathcal{V} \setminus \bigcup_{i=1}^{n} a_i$. A VENN REGION of D is either a proper Venn region of D or the non-proper Venn region of D.

Definition 23. (Places). Let V be a collection of variables. A PLACE of V is any *non-null* map $\pi : V \to \{0, 1\}$.

Thus, if $\mathcal{A}$ is an assignment over a collection of variables $V = \{x_1, \ldots, x_n\}$, for each $\emptyset \neq \alpha \subseteq \{1, \ldots, n\}$, the proper region $\sigma = \bigcap_{i\in\alpha} x_i^{\mathcal{A}} \setminus \bigcup_{j\notin\alpha} x_j^{\mathcal{A}}$ of the Venn diagram $\{x^{\mathcal{A}} : x \in V\}$ can be conveniently represented by the place π_σ of V defined by putting

$$\pi_\sigma(x_i) = \begin{cases} 1 & \text{if } i \in \alpha \\ 0 & \text{otherwise.} \end{cases}$$

It is to be noticed that in the literature the notion of place has usually been defined relative to a given conjunction $\mathcal{C}$ of normalized literals. In particular, a place of $\mathcal{C}$ has been characterized by requiring that, when considered as a truth assignment over the variables of $\mathcal{C}$ (regarded as propositional letters), it satisfies the following associated propositional conjunction $\mathcal{C}_P$

$$\begin{aligned} \mathcal{C}_P \;=\; & \{x \leftrightarrow y : \text{ the literal } x = y \text{ is in } \mathcal{C}\} \cup \\ & \{x \leftrightarrow (y \vee z) : \text{ the literal } x = y \cup z \text{ is in } \mathcal{C}\} \cup \\ & \{x \leftrightarrow (y \wedge \neg z) : \text{ the literal } x = y \setminus z \text{ is in } \mathcal{C}\} \end{aligned}$$

(cf. conditions (C1)–(C3) of Definition 25 below).

The way in which places are used to represent models of set-theoretic formulae is illustrated by the following elementary case. Let $\mathcal{C}$ be a conjunction of set literals of the form

$$x = y, \quad x \neq y, \quad x = y \cup z, \quad x = y \setminus z, \tag{6}$$

and let V be the collection of variables occurring in $\mathcal{C}$. Given a set model $\mathcal{M}$ for $\mathcal{C}$, we can construct its *syntactic representation* as the collection of places $\Pi_{\mathcal{C},\mathcal{M}}$ corresponding to the non-empty proper regions of the Venn diagram $\{x^{\mathcal{M}} : x \in V\}$. It can easily be checked that the following *adequacy conditions* are then verified

(i) each place $\pi \in \Pi_{\mathcal{C},\mathcal{M}}$ satisfies the associated propositional conjunction $\mathcal{C}_P$, when considered as a truth assignment over V; and
(ii) for every literal of type $x \neq y$ in $\mathcal{C}$, there exists a place $\pi \in \Pi_{\varphi,\mathcal{M}}$ such that $\pi(x) \neq \pi(y)$, i.e. π satisfies the propositional formula $\neg(x \leftrightarrow y)$ (cf. condition (C4) of Definition 25 below).

Moreover, as will be shown below, with any system of places satisfying the above conditions (i) and (ii) we can associate a corresponding *canonical assignment* which satisfies $\mathcal{C}$. Therefore, since the existence of such systems can be checked effectively, we have a decision procedure for conjunctions of literals of type (6).

5.2. Place frameworks, adequacy conditions, and syntactic descriptions.

When in addition to literals of type (6) our conjunction $\mathcal{C}$ contains also literals of type $x = \{y\}$, i.e. $\mathcal{C}$ is a conjunction of normalized **MLSS**-literals, the adequacy conditions become more involved and they are better stated in terms also of a *variables map* denoted by at. It turns out that the map at needs to be defined only over *singleton variables*, namely those variables which occur in terms of type $\{x\}$ in $\mathcal{C}$. For such variables, $at(x)$ is intended to denote the place corresponding to the Venn region which contains $x^{\mathcal{C}}$.

Systems of places are therefore generalized to *place frameworks* according to the following definition.

Definition 24. (Place frameworks). A PLACE FRAMEWORK is a quadruple (V, W, Π, at) such that

- V is a collection of variables;
- W is a subset of V;
- Π is a collection of places of V;
- at is a map from W into Π.

Then the adequacy conditions can be stated as follows.

Definition 25. (Adequacy conditions). Let $\mathcal{C}$ be a conjunction of normalized **MLSS**-literals, let V be the collection of variables occurring in $\mathcal{C}$, and let W be the collection of variables occurring in terms of type $\{x\}$ in $\mathcal{C}$, called SINGLETON VARIABLES. A place framework (V, W, Π, at) is ADEQUATE for $\mathcal{C}$ if it satisfies the following conditions:

(C1) if $x = y$ is in $\mathcal{C}$, then $\pi(x) = \pi(y)$, for each $\pi \in \Pi$;
(C2) if $x = y \cup z$ is in $\mathcal{C}$, then $\pi(x) = 1$ if and only if $\pi(y) = 1$ or $\pi(z) = 1$, for each $\pi \in \Pi$;

(C3) if $x = y \setminus z$ is in $\mathcal{C}$, then $\pi(x) = 1$ if and only if $\pi(y) = 1$ and $\pi(z) = 0$, for each $\pi \in \Pi$;
(C4) if $x \neq y$ is in $\mathcal{C}$, then $\pi(x) \neq \pi(y)$, for some $\pi \in \Pi$;
(C5) if $x = \{y\}$ is in $\mathcal{C}$, then
(C5.1) $at(y)$ is the unique place $\pi \in \Pi$ such that $\pi(x) = 1$;
(C5.2) if $at(z) = at(y)$, for some variable $z \in W$, then $\pi(z) = \pi(y)$, for every $\pi \in \Pi$;
(C5.3) if $\mathcal{C}$ contains a literal $x_1 = \{y_1\}$ such that $\pi(y_1) = \pi(y)$ for every $\pi \in \Pi$, then $at(y_1) = at(y)$;
(C6) the relation $\prec$ over Π is acyclic, where $\prec$ is defined by putting $\pi_1 \prec \pi_2$ if and only if there exists a literal of the form $x = \{y\}$ in $\mathcal{C}$ such that $\pi_1(y) = \pi_2(x) = 1$.

As already observed, (C1)–(C3) correspond to the condition (i) stated in the previous section, whereas (C4) is just condition (ii). Condition (C5) characterizes syntactically literals of type $x = \{y\}$. Finally, condition (C6) is the syntactic counterpart of the axiom of regularity, which forbids membership cycles (cf. [12]).

Any given assignment can suitably be represented by means of its syntactic description, according to the following definition.

Definition 26. (Syntactic descriptions). Let $\mathcal{A}$ be a set assignment defined over a collection V of variables, and let W be a subset of V.

We define the COLLECTION OF PLACES Π and Variables Map $at : W \to \Pi$ associated with $\mathcal{A}$, V, and W as follows.

Let Σ be the collection of the non-empty proper regions of the Venn diagram $\{x^{\mathcal{A}} : x \in V\}$. With each $\sigma \in \Sigma$ we associate a place $\pi_\sigma : V \to \{0, 1\}$ such that $\pi_\sigma(x) = 1$ if and only if $\sigma \subseteq x^{\mathcal{A}}$, for every variable $x \in V$. Then we put

$$\Pi = \{\pi_\sigma : \sigma \in \Sigma\}.$$

In addition, for each variable $w \in W$ we denote by σ_w the unique region of the Venn diagram $\{x^{\mathcal{A}} : x \in V\}$ such that $w^{\mathcal{C}} \in \sigma_w$. Then we put

$$at(w) = \pi_{\sigma_w}, \qquad \text{for } w \in W.$$

The SYNCTATIC DESCRIPTION OF $\mathcal{A}$ RELATIVE TO V AND W is the place framework (V, W, Π, at), where Π and at are respectively the collection of places and variables map associated with $\mathcal{A}$, V, and W.

It turns out that the syntactic description of a model of a given conjunction $\mathcal{C}$ of normalized **MLSS**-literals is a place framework which is adequate for $\mathcal{C}$. This is proven in the following lemma.

Lemma 27. *Let $\mathcal{C}$ be a conjunction of normalized* **MLSS**-*literals, let V be the collection of variables occurring in $\mathcal{C}$, and let W be the collection of singleton variables of $\mathcal{C}$, namely variables occurring in terms of type $\{x\}$ in $\mathcal{C}$. Let us assume that $\mathcal{C}$ is satisfiable and let $\mathcal{A}$ be a model of $\mathcal{C}$. Then the syntactic description (V, W, Π, at) of $\mathcal{A}$ relative to V and W is adequate for $\mathcal{C}$.*

Proof. Let $\mathcal{C}$, V, W, $\mathcal{A}$ be as in the hypotheses, and let (V, W, Π, at) be the syntactic description of $\mathcal{A}$ relative to V and W. We prove that (V, W, Π, at) is adequate for $\mathcal{C}$ by verifying that all conditions (C1)–(C6) in Definition 25 are satisfied.

Concerning (C1), let the literal $x = y$ be in $\mathcal{C}$, and assume that $\pi_\sigma(x) = 1$, for some non-empty region σ of the Venn diagram $\{x^{\mathcal{A}} : x \in V\}$. Then $\sigma \subseteq x^{\mathcal{A}} = y^{\mathcal{A}}$, implying $\pi_\sigma(y) = 1$. Conditions (C2) and (C3) can be verified much in the same way.

To verify (C4), let the literal $x \neq y$ be in $\mathcal{C}$. Then $x^{\mathcal{A}} \neq y^{\mathcal{A}}$ and therefore there must exist a non-empty Venn region σ such that $\sigma \subseteq (x^{\mathcal{A}} \setminus y^{\mathcal{A}}) \cup (y^{\mathcal{A}} \setminus x^{\mathcal{A}})$. Thus, $\pi_\sigma(x) \neq \pi_\sigma(y)$ follows.

Concerning (C5), let the literal $x = \{y\}$ be in $\mathcal{C}$. Assume first that $\pi(x) = 1$. Then there exists a non-empty Venn region σ such that $\pi = \pi_\sigma$ and $\sigma \subseteq x^{\mathcal{A}}$. But then $\sigma = \{y^{\mathcal{A}}\}$ and therefore $at(y) = \pi_\sigma = \pi$. On the other hand, if $at(y) = \pi$, then there exists a non-empty Venn region σ such that $y^{\mathcal{A}} \in \sigma$ and $\pi_\sigma = \pi$. But then $\sigma \subseteq x^{\mathcal{A}}$, implying $\pi_\sigma(x) = 1$, namely $\pi(x) = 1$, thus proving (C5.1). Next, let $at(z) = at(y)$, for some $z \in V$. Then, $z^{\mathcal{A}}, y^{\mathcal{A}} \in \sigma \subseteq x^{\mathcal{A}}$, for some Venn region σ. Hence $z^{\mathcal{A}} = y^{\mathcal{A}}$, so that it follows immediately that $\pi(z) = \pi(y)$, for every $\pi \in \Pi$, namely condition (C5.2) holds.

Finally, assume that $\mathcal{C}$ contains a literal $x_1 = \{y_1\}$ such that $\pi(y_1) = \pi(y)$, for every $\pi \in \Pi$. Then $y_1^{\mathcal{A}}$ and $y^{\mathcal{A}}$ contain the same non-empty Venn regions, so that they are identical. Hence $at(y_1) = at(y)$, proving that condition (C5.3) holds.

To show that also condition (C6) is satisfied, it is enough to verify that $\pi_{\sigma'} \prec \pi_{\sigma''}$ implies $rank(\sigma') < rank(\sigma'')$, for any two non-empty Venn regions σ' and σ'' of $\{x^{\mathcal{A}} : x \in V\}$. Indeed, if this is the case, then a cycle $\pi_1 \prec \pi_2 \prec \ldots \prec \pi_m \prec \pi_1$ in Π would entail a cycle $rank(\sigma_1) < rank(\sigma_2) < \ldots < rank(\sigma_m) < rank(\sigma_1)$, where the σ_i's are the Venn region corresponding to the places π_i's, for $i = 1, \ldots, m$, a contradiction. Thus, let $\pi_{\sigma'} \prec \pi_{\sigma''}$, for two given non-empty Venn regions σ' and σ''. Then there exists a literal of the form $x = \{y\}$ such that $\pi_{\sigma'}(y) = \pi_{\sigma''}(x) = 1$, so that $\sigma' \subseteq y^{\mathcal{A}} \in x^{\mathcal{A}} = \sigma''$. Therefore, properties (i) and (ii) of the function $rank$ listed in Subsection 2.1 yield $rank(\sigma') < rank(\sigma'')$. $\square$

5.3. Canonical assignments.

Next we show how we can construct canonical assignments in correspondence of a place framework which is adequate for a given conjunction $\mathcal{C}$ of normalized **MLSS**-literals. This will be done in such a way that the resulting canonical assignments will satisfy $\mathcal{C}$.

Definition 28. (Canonical assignments). Let $\mathcal{C}$ be a conjunction of normalized **MLSS**-literals and let (V, W, Π, at) be a place framework which is *adequate* for $\mathcal{C}$. Let $\prec$ be the relation over Π defined by putting $\pi_1 \prec \pi_2$ if and only if there exists a literal of the form $x = \{y\}$ in $\mathcal{C}$ such that $\pi_1(y) = \pi_2(x) = 1$. Let $\{u_\pi : \pi \in \Pi \setminus range(at)\}$ be any collection of sets such that

- $u_\pi \neq u_{\pi'}$, for every two distinct places $\pi, \pi' \in \Pi \setminus range(at)$;
- $|u_\pi| \geq |\Pi|$, for every $\pi \in \Pi \setminus range(at)$.

Using the chosen sets u_π's and following any linear ordering which extends the relation $\prec$, whose existence is ensured by condition (C6) of Definition 25, we put

$$\overline{\pi} = \begin{cases} \{u_\pi\}, & \text{if } \pi \notin range(at), \\ \{\bigcup\limits_{\pi' \prec \pi} \overline{\pi}'\}, & \text{if } \pi \in range(at), \end{cases}$$

for each place $\pi \in \Pi$. The resulting map $\pi \mapsto \overline{\pi}$ is called a PLACE REALIZATION OF Π.

Finally, we define an assignment $\mathcal{M}$ over V by putting

$$x^{\mathcal{M}} = \bigcup_{\pi(x)=1} \overline{\pi},$$

for each variable $x \in V$. The assignment $\mathcal{M}$ is called a CANONICAL ASSIGNMENT RELATIVE TO (V, W, Π, at) (and $\mathcal{C}$).

As anticipated, canonical assignments relative to a given conjunction $\mathcal{C}$ are models of $\mathcal{C}$.

Lemma 29. *Let $\mathcal{C}$ be a conjunction of normalized* **MLSS**-*literals over a set $V = \{x_1, \ldots, x_n\}$ of variables and let $\mathcal{M}$ be a canonical assignment relative to a given place framework which is adequate for $\mathcal{C}$. Then $\mathcal{M}$ satisfies $\mathcal{C}$.*

Proof. Let (V, W, Π, at) be a place framework which is adequate for a given conjunction $\mathcal{C}$ of normalized **MLSS**-literals and let $\{u_\pi : \pi \in \Pi \setminus range(at)\}$ be a collection of sets such that

- $u_\pi \neq u_{\pi'}$, for every two distinct places $\pi, \pi' \in \Pi \setminus range(at)$;

– $|u_\pi| \geq |\Pi|$, for every $\pi \in \Pi \setminus \mathit{range}(at)$.

Let $\pi \mapsto \overline{\pi}$ be the place realization of Π relative to $\{u_\pi : \pi \in \Pi \setminus \mathit{range}(at)\}$ and let $\mathcal{M}$ be the canonical assignment relative to $\{u_\pi : \pi \in \Pi \setminus \mathit{range}(at)\}$ defined by

$$x^{\mathcal{M}} = \bigcup_{\pi(x)=1} \overline{\pi},$$

for each variable $x \in V$.

We intend to show that $\mathcal{M}$ is a model of $\mathcal{C}$, whose non-empty proper Venn regions are just the sets $\overline{\pi}$'s.

Thus, we show first that the sets $\overline{\pi}$'s are non-empty and pairwise disjoint. By construction, it is immediate to see that they are all singletons, so in particular they are non-empty. To show that they are also pairwise disjoint, we proceed by contradiction. Thus, let $\overline{\pi}' \cap \overline{\pi}'' \neq \emptyset$, for some distinct places $\pi', \pi'' \in \Pi$. Let $\prec^+$ be any linear ordering of Π which extends $\prec$, where $\prec$ is the relation over Π defined by putting $\pi_1 \prec \pi_2$ if and only if there exists a literal of the form $x = \{y\}$ in $\mathcal{C}$ such that $\pi_1(y) = \pi_2(x) = 1$. Let (π_1, π_2) be the ordered pair in Π^2 such that

(a) $\overline{\pi}_1 \cap \overline{\pi}_2 \neq \emptyset$;
(b) $\overline{\pi}_1 \prec^+ \overline{\pi}_2$;
(c) the sets $\overline{\pi}$'s such that $\pi \prec^+ \pi_2$ are pairwise disjoint.

We must consider the following cases:

– $|\{\overline{\pi}_1, \overline{\pi}_2\} \cap \mathit{range}(at)| = 0$;
– $|\{\overline{\pi}_1, \overline{\pi}_2\} \cap \mathit{range}(at)| = 1$;
– $|\{\overline{\pi}_1, \overline{\pi}_2\} \cap \mathit{range}(at)| = 2$.

The first case is ruled out by the very definition of the sets $\overline{\pi}$'s, since by construction all u_π's are taken pairwise distinct. The second case is ruled out by observing that the cardinality of the element of $\overline{\pi}'$ is greater than or equal to $|\Pi|$, for each $\pi' \notin \mathit{range}(at)$, whereas the cardinality of the element of $\overline{\pi}''$ is strictly less than $|\Pi|$, for each $\pi'' \in \mathit{range}(at)$. Thus, we may assume that $\overline{\pi}_1, \overline{\pi}_2 \in \mathit{range}(at)$. Let $x_1 = \{y_1\}$ and $x_2 = \{y_2\}$ be literals in $\mathcal{C}$ such that $\pi_1 = at(y_1)$ and $\pi_2 = at(y_2)$. By (a), we have $\bigcup_{\pi' \prec \pi_1} \overline{\pi}' = \bigcup_{\pi'' \prec \pi_2} \overline{\pi}''$. Hence, by (b) and (c), we have that

$$\pi \prec \pi_1 \text{ if and only if } \pi \prec \pi_2, \text{ for every } \pi \in \Pi.$$

Therefore, (C5.1) and (C5.2) imply that

$$\pi(y_1) = \pi(y_2), \text{ for every } \pi \in \Pi,$$

so that, by (C5.3), we have $at(y_1) = at(y_2)$, namely $\pi_1 = \pi_2$ holds, which is a contradiction.

We prove next that $\mathcal{M}$ is a model of $\mathcal{C}$ by showing that it satisfies all literals in $\mathcal{C}$.

Literals of the form $x = y$. Let the literal $x = y$ be in $\mathcal{C}$. Then, by (C1), $x^{\mathcal{M}} = \bigcup_{\pi(x)=1} \overline{\pi} = \bigcup_{\pi(y)=1} \overline{\pi} = y^{\mathcal{M}}$.

Literals of the form $x = y \cup z$ *and* $x = y \setminus z$. These cases are similar to the case of literals of the form $x = y$, but using properties (C2) and (C3) in place of (C1).

Literals of the form $x \neq y$. Let the literal $x \neq y$ be in $\mathcal{C}$. Then, by (C4), there exists a place π_0 such that $\pi_0(x) \neq \pi_0(y)$. Since all sets $\overline{\pi}$'s are non-empty and pairwise disjoint, we have $\emptyset \neq \overline{\pi}_0 \subseteq (x^{\mathcal{M}} \setminus y^{\mathcal{M}}) \cup (y^{\mathcal{M}} \setminus x^{\mathcal{M}})$, proving that $x^{\mathcal{M}} \neq y^{\mathcal{M}}$.

Literals of the form $x = \{y\}$. Let the literal $x = \{y\}$ be in $\mathcal{C}$, and let $\pi_0 = at(y)$. Then, by (C5.1), π_0 is the unique place $\pi \in \Pi$ such that $\pi(x) = 1$, so that $x^{\mathcal{M}} = \overline{\pi}_0 = \{\bigcup_{\pi \prec \pi_0} \overline{\pi}\}$. Thus, in order to prove that $x^{\mathcal{M}} = \{y^{\mathcal{M}}\}$ it is enough to show that $\bigcup_{\pi \prec \pi_0} \overline{\pi} = \bigcup_{\pi(y)=1} \overline{\pi}$. Since the sets $\overline{\pi}$'s are non-empty and pairwise disjoint, this amount to show that $\pi \prec \pi_0$ if and only if $\pi(y) = 1$, for every $\pi \in \Pi$. To this end, let $\pi(y) = 1$. Then, by the very definition of the relation $\prec$ we have that $\pi \prec \pi_0$. Conversely, if $\pi \prec \pi_0$, then there exists a literal of the form $u = \{v\}$ in $\mathcal{C}$ such that $\pi_0(u) = \pi(v) = 1$. Therefore $\pi_0 = at(v)$, so that (C5.2) yields $\pi(y) = 1$.

This concludes the proof of the lemma. $\square$

5.4. Singleton model property and decidability of MLSS.

Lemmas 27 and 29 allow us to establish easily the singleton model property for conjunctions of normalized **MLSS**-literals. Indeed, let $\mathcal{C}$ be such a conjunction over a set of variables V and assume that it is satisfiable by a model $\mathcal{A}$. Then, from Lemma 27, the syntactic description $\mathfrak{B}$ of $\mathcal{A}$ is a place framework which is adequate for $\mathcal{C}$. Hence, by Lemma 29, any canonical assignment $\mathcal{M}$ relative to $\mathfrak{B}$ satisfies $\mathcal{C}$ and since, by construction, all proper regions of the Venn diagram $\{x^{\mathcal{M}} : x \in V\}$ have cardinality at most 1, we are done. Thus, we have:

Theorem 30. (Singleton model property). *Let $\mathcal{C}$ be a satisfiable conjunction of normalized* **MLSS**-*literals over a set $V = \{x_1, \ldots, x_n\}$ of variables. Then there exists a model $\mathcal{M}$ of $\mathcal{C}$ such that*

$$\left| \bigcap_{i \in \alpha} x_i^{\mathcal{M}} \setminus \bigcup_{j \notin \alpha} x_j^{\mathcal{M}} \right| \leq 1, \qquad \textit{for each } \alpha \in \mathcal{P}^+(\{1, \ldots, n\}).$$

We wish to stress the fact that the result of Theorem 30 is specific to conjunctions of *normalized* **MLSS**-literals of the form (4). Other **MLSS**-formulae do not enjoy necessarily the singleton model property. Consider for instance the following satisfiable conjunction of **MLS**-literals:

$$\mathcal{C} = \{x_1 \in y,\ x_2 \in y,\ x_1 \neq x_2,\ x_1 \notin x_2,\ x_2 \notin x_1\}.$$

Then, it is easy to see that the Venn region $y^{\mathcal{M}} \setminus (x_1^{\mathcal{M}} \cup x_2^{\mathcal{M}})$ cannot be a singleton set, for any model $\mathcal{M}$ of $\mathcal{C}$, since it must contain at least the two distinct elements $x_1^{\mathcal{M}}$ and $x_2^{\mathcal{M}}$. In fact, it could be shown that despite the fact that **MLS** is a sublanguage of **MLSS**, it does not enjoy the singleton model property, regardless of the normalization adopted.

Lemmas 27 and 29 can also be used to show the decidability of **MLSS**. Indeed, let $\mathcal{C}$ be a conjunction of normalized **MLSS**-literals. If $\mathcal{C}$ is satisfiable, then it has a model $\mathcal{A}$, so that, by Lemma 27, the syntactic description of $\mathcal{A}$ is a place framework which is adequate for $\mathcal{C}$. Conversely, if there exists a place framework $\mathfrak{B}$ which is adequate for $\mathcal{C}$, then, by Lemma 29, any canonical assignment relative to $\mathfrak{B}$ satisfies $\mathcal{C}$. Thus, we have proven

Lemma 31. *A conjunction of normalized* **MLSS**-*literals is satisfiable if and only if there exists a place framework which is adequate for it.*

Lemmas 31 and 10, and the fact that the collection of all possible place frameworks (V, W, Π, at) is finite, for any finite sets of variables V and W, yield immediately the decidability of **MLSS**.

Theorem 32. (Decidability of MLSS). *The satisfiability problem for* **MLSS** *is decidable.*

5.5. An application of the singleton model property.

Our final goal is to prove Lemma 19, which states that if $\mathcal{C}$ is a conjunction of normalized **MLSSmf**-literals over the collection of variables $V = \{x_1, \dots, x_n\}$ and $\mathcal{C}^*$ is the result of applying to $\mathcal{C}$ the translation algorithm in Figure 1, then there exists a model $\mathcal{B}$ of $\mathcal{C}^*$ such that $|v_\alpha^{\mathcal{B}}| \leq 1$, for each $\alpha \in \mathcal{P}^+(\{1, \dots, n\})$, provided that $\mathcal{C}^*$ is satisfiable. We recall that, loosely speaking, the variables v_α, for $\alpha \in \mathcal{P}^+(\{1, \dots, n\})$, are constrained so as to represent the proper Venn regions of $x_1, \dots, x_n$.

It could therefore be expected that Lemma19 is a direct consequence of the singleton model property for conjunctions of normalized **MLSS**-literals proven in Theorem 30. Unfortunately, this is not the case, since in general

- $\mathcal{C}^*$ can contain non-normalized conjuncts;
- the sets $v_\alpha^{\mathcal{B}}$'s are not proper regions of the Venn diagram $\{x^{\mathcal{B}} : x \text{ occurs in } \mathcal{C}^*\}$, but of the *smaller* Venn diagram $\{x^{\mathcal{B}} : x \in V\}$.

A major step towards the proof of Lemma 19 is the following technical result.

Lemma 33. *Let $\mathcal{C}$ be a conjunction of normalized* **MLSS**-*literals over a collection V of variables, and let U be a subset of V. Assume that there exists a model $\mathcal{B}$ of $\mathcal{C}$ such that:*

(a) $u^{\mathcal{B}} \cap v^{\mathcal{B}} = \emptyset$, *for any two distinct variables $u, v \in U$;*
(b) *if a literal of the form $x \neq y$ or of the form $x = \{y\}$ is in $\mathcal{C}$, then there are subsets L, M of U such that* $x^{\mathcal{B}} = \bigcup_{u \in L} u^{\mathcal{B}}$ *and* $y^{\mathcal{B}} = \bigcup_{u \in M} u^{\mathcal{B}}$.

Then there exists a model $\mathcal{M}$ of $\mathcal{C}$ such that $|u^{\mathcal{M}}| \leq 1$, for each $u \in U$.

Proof. Let $\mathcal{B}$ be a model for $\mathcal{C}$ satisfying properties (a) and (b) of the lemma. Also, let $\mathfrak{B} = (V, W, \Pi, at)$ be the syntactic description of the assignment $\mathcal{B}$ relative to V and W, where W is the collection of singleton variables occurring in $\mathcal{C}$ (cf. Definition 25). By Lemma 27, the place framework $\mathfrak{B}$ is adequate for $\mathcal{C}$.

Our aim is to exploit properties (a) and (b) in order to define another place framework $\mathfrak{B}' = (V, W, \Pi', at')$ which is adequate for $\mathcal{C}$ and such that

($\star$) for each $u \in U$, there exists at most one place $\pi \in \Pi'$ such that $\pi(u) = 1$.

Indeed, by Lemma 29 and ($\star$), it would easily follow that any canonical assignment $\mathcal{M}$ relative to $\mathfrak{B}'$ and $\mathcal{C}$ would be a model of $\mathcal{C}$ such that $|u^{\mathcal{M}}| \leq 1$, for each $u \in U$.

We begin by proving the following property.

Fact 1. *There are no two distinct variables $u, v \in U$ such that $\pi(u) = \pi(v) = 1$, for any place $\pi \in \Pi$.*

Proof. Let us assume by contradiction that $\pi(u) = \pi(v) = 1$, for some distinct variables $u, v \in U$ and place $\pi \in \Pi$, and let σ be the region of the Venn diagram $\{x^{\mathcal{B}} : x \in V\}$ corresponding to the place π. Then we would have $\emptyset \neq \sigma \subseteq u^{\mathcal{B}} \cap v^{\mathcal{B}}$, contradicting the hypothesis $u^{\mathcal{B}} \cap v^{\mathcal{B}} = \emptyset$. □

Next we define a relation $\sim$ over Π by putting

$$\pi_1 \sim \pi_2 \quad \text{if and only if} \quad \pi_1 = \pi_2 \text{ or } \pi_1(u) = \pi_2(u) = 1, \text{ for some } u \in U.$$

We claim that $\sim$ is an equivalence relation. Clearly $\sim$ is reflexive and symmetric. For the transitivity, let us assume that $\pi_1 \sim \pi_2$ and $\pi_2 \sim \pi_3$. If either $\pi_1 = \pi_2$ or $\pi_2 = \pi_3$, then we have plainly $\pi_1 \sim \pi_3$; otherwise there must exist variables $u, v \in U$ such that $\pi_1(u) = \pi_2(u) = 1$ and $\pi_2(v) = \pi_3(v) = 1$. Since $\pi_2(u) = \pi_2(v) = 1$, Fact 1 implies that u and v must coincide. Therefore we have $\pi_1(u) = \pi_3(u) = 1$, proving that $\pi_1 \sim \pi_3$.

Notice that in general $\pi_1 \sim \pi_2$ does not imply that $\pi_1(x) = \pi_2(x)$, for every variable $x \in V$. Nevertheless, if $\pi_1 \sim \pi_2$ then $\pi_1(u) = \pi_2(u)$ holds, for every variable $u \in U$. In fact, we can prove the following stronger property.

Fact 2. *If $\pi_1 \sim \pi_2$, then π_1 and π_2 agree on all variables $x \in V$ such that $x^{\mathcal{B}} = \bigcup_{u \in L} u^{\mathcal{B}}$, for some subset L of U.*

Proof. Let $\pi_1 \sim \pi_2$ and let $x \in V$ be such that $x^{\mathcal{B}} = \bigcup_{u \in L} u^{\mathcal{B}}$, for some $L \subseteq U$. The proposition is trivial if $\pi_1 = \pi_2$. Thus, let us assume that $\pi_1 \neq \pi_2$, so that there must exist a variable $u_0 \in U$ such that $\pi_1(u_0) = \pi_2(u_0) = 1$. Let σ be the non-empty region of the Venn diagram $\{y^{\mathcal{B}} : y \in V\}$ corresponding to π_1. We have $\sigma \subseteq u_0^{\mathcal{B}}$ and therefore the pairwise disjointness of the sets $u^{\mathcal{B}}$ yields that σ is a subset of $x^{\mathcal{B}}$ if and only if $u_0 \in L$, i.e., $\pi_1(x) = 1$ if and only if $u_0 \in L$. Likewise, we have $\pi_2(x) = 1$ if and only if $u_0 \in L$. Hence, π_1 and π_2 agree on x. □

For each $\pi \in \Pi$, we denote by $[\pi]\sim$ the equivalence class of $\sim$ containing π. Also, we denote by $\Pi/\sim$ the set of all equivalence classes of $\sim$.

In order to define a new place framework adequate for $\mathcal{C}$ and satisfying ($\star$), we fix a choice map $\cdot^r$ from $\Pi/\sim$ into Π such that

$$\mathfrak{p}^r \in \mathfrak{p}, \qquad \text{for every } \mathfrak{p} \in \Pi/\sim,$$

and we define $\Pi' \subseteq \Pi$ and $at' : W \to \Pi'$ as follows:

$$\begin{aligned} \Pi' &= \{[\pi]^r_\sim : \pi \in \Pi\}, \\ at'(y) &= [at(y)]^r_\sim, \qquad \text{for each } y \in W. \end{aligned}$$

Next we show that the place framework $\mathfrak{B}' = (V, W, \Pi', at')$ is adequate for $\mathcal{C}$, by verifying that it satisfies conditions (C1)–(C6) of Definition 25. We will use the fact that, as observed at the beginning of the proof, the syntactic description $\mathfrak{B} = (V, W, \Pi, at)$ of $\mathcal{B}$ (from which $\mathfrak{B}'$ has been derived) is adequate for $\mathcal{C}$.

Conditions (C1)–(C3) are plainly satisfied by $\mathfrak{B}'$, since they are satisfied by $\mathfrak{B}$ and $\Pi' \subseteq \Pi$.

Concerning (C4), let the literal $x \neq y$ be in $\mathcal{C}$. Then there exists $\pi_0 \in \Pi$ such that $\pi_0(x) \neq \pi_0(y)$. By (b), there exist $L, M \subseteq U$ such that $x^{\mathcal{B}} = \bigcup_{u \in L} u^{\mathcal{B}}$ and $y^{\mathcal{B}} = \bigcup_{u \in M} u^{\mathcal{B}}$. Since $[\pi_0]^r_\sim \sim \pi_0$, Fact 2 implies that $[\pi_0]^r_\sim(x) = \pi_0(x)$ and $[\pi_0]^r_\sim(y) = \pi_0(y)$, so that $[\pi_0]^r_\sim(x) \neq [\pi_0]^r_\sim(y)$.

In order to verify condition (C5), let us assume that the literal $x = \{y\}$ occurs in $\mathcal{C}$. Then, by condition (b) of the lemma, we have $x^{\mathcal{B}} = \bigcup_{u \in L} u^{\mathcal{B}}$ and $y^{\mathcal{B}} = \bigcup_{u \in M} u^{\mathcal{B}}$, for some $L, M \subseteq U$. We first consider subcondition (C5.1). By the adequacy of $\mathfrak{B}$, $at(y)$ is the unique place $\pi \in \Pi$ such that $\pi(x) = 1$. Let $\pi' \in \Pi'$ be such that $\pi'(x) = 1$. Since $\Pi' \subseteq \Pi$, we have $\pi' = at(y)$, so that $\pi' = [\pi']^r_\sim = [at(y)]^r_\sim = at'(y)$. Conversely, since $at'(y) \sim at(y)$, by Fact 2 we have $(at'(y))(x) = (at(y))(x) = 1$.

Concerning subcondition (C5.2), let $at'(z) = at'(y)$, for some singleton variable $z \in W$. Then $[at(z)]^r_\sim = [at(y)]^r_\sim$, so that $at(z) \sim at(y)$. If $at(z) = at(y)$, then, by the adequacy of $\mathfrak{B}$, we have $[\pi]^r_\sim(z) = [\pi]^r_\sim(y)$, for every $\pi \in \Pi$. On the other hand, if $(at(z))(u_0) = (at(y))(u_0) = 1$, for some $u_0 \in U$, then $z^{\mathcal{B}} \in u_0^{\mathcal{B}}$ and $y^{\mathcal{B}} \in u_0^{\mathcal{B}}$. Moreover, since $\{y^{\mathcal{B}}\} = x^{\mathcal{B}} = \bigcup_{u \in L} u^{\mathcal{B}}$, by condition (a) of the lemma we have $x^{\mathcal{B}} = u_0^{\mathcal{B}}$, so that $y^{\mathcal{B}} = z^{\mathcal{B}}$. Therefore, we have again $[\pi]^r_\sim = [\pi]^r_\sim(y)$, for every $\pi \in \Pi$.

For subcondition (C5.3), let $x_1 = \{y_1\}$ be a literal occurring in $\mathcal{C}$ such that $[\pi]^r_\sim(y_1) = [\pi]^r_\sim(y)$, for every $\pi \in \Pi$. By condition (b) of the lemma, $y_1^{\mathcal{B}} = \bigcup_{u \in M_1} u^{\mathcal{B}}$, for some $M_1 \subseteq U$. Then, since $\pi \sim [\pi]^r_\sim$, Fact 2 implies that $\pi(y_1) = [\pi]^r_\sim(y_1) = [\pi]^r_\sim(y) = \pi(y)$, for every $\pi \in \Pi$. Therefore $y_1^{\mathcal{B}} = y^{\mathcal{B}}$, so that $at(y_1) = at(y)$ and *a fortiori* we have $at'(y_1) = [at(y_1)]^r_\sim = [at(y)]^r_\sim = at'(y)$.

Finally, concerning condition (C6), let $\prec'$ be the relation on Π' defined by putting $\pi'_1 \prec' \pi'_2$ if and only if there exists a literal $x = \{y\}$ in $\mathcal{C}$ such that $\pi'_1(y) \prec' \pi'_2(x)$, for every $\pi'_1, \pi'_2 \in \Pi'$. Since $\prec'$ is the restriction on Π' of the analogous relation $\prec$ relative to the place framework $\mathfrak{B}$, from the adequacy of $\mathfrak{B}$ it plainly follows that $\prec'$ is acyclic too.

Summing up, we have proven that the place framework $\mathfrak{B}' =$

(V, W, Π', at') is adequate for $\mathcal{C}$.

Let $\mathcal{M}$ be any canonical assignment relative to the place framework $\mathfrak{B}'$ and let $\pi \mapsto \overline{\pi}$ be its underlying place realization, so that

$$x^{\mathcal{M}} = \bigcup_{\substack{\pi(x)=1 \\ \pi\in\Pi'}} \overline{\pi}, \qquad \text{for each } x \in V.$$

By Lemma 29, our conjunction $\mathcal{C}$ is satisfied by $\mathcal{M}$.

In order to complete the proof of the lemma, it only remains to show that $|u^{\mathcal{M}}| \leq 1$, for every $u \in U$. Thus, let $u \in U$. Since all sets $\overline{\pi}$'s are pairwise disjoint singletons, it is enough to show that $u^{\mathcal{M}} \subseteq \overline{\pi}'$, for some $\pi' \in \Pi'$, i.e. there is at most one place $\pi' \in \Pi'$ such that $\pi'(u) = 1$. Indeed, let us assume that $[\pi_1]^r_\sim(u) = [\pi_2]^r_\sim(u) = 1$, for some $\pi_1, \pi_2 \in \Pi$. Then, by the very definition of $\sim$, we have $\pi_1 \sim [\pi_1]^r_\sim \sim [\pi_2]^r_\sim \sim \pi_2$, so that $[\pi_1]_\sim = [\pi_2]_\sim$, and therefore $[\pi_1]^r_\sim = [\pi_2]^r_\sim$.

Hence the lemma is proven. $\square$

We finally have everything we need to prove Lemma 19. For convenience, we restate it.

Lemma 2. *Let $\mathcal{C}$ be a conjunction of normalized* **MLSSmf***-literals over the collection of variables $V = \{x_1, \ldots, x_n\}$, and let $\mathcal{C}^*$ be the result of applying the translation algorithm in Figure 1 to $\mathcal{C}$. Then, if $\mathcal{C}^*$ is satisfiable, there exists a model $\mathcal{B}$ of $\mathcal{C}$ such that $|v_\alpha^{\mathcal{B}}| \leq 1$, for each $\alpha \in \mathcal{P}^+(\{1, \ldots, n\})$.*

Proof. Let us assume that $\mathcal{C}^*$ is satisfiable, and let $\varphi = \psi_1 \vee \ldots \vee \psi_k$ be a disjunctive normal form of $\mathcal{C}^*$. By suitably adding new variables, it is possible to construct a disjunction $\varphi' = \psi_1' \vee \ldots \vee \psi_k'$ equisatisfiable with φ and such that ψ_i' is a conjunction of normalized **MLSS**-literals, for each $i = 1, \ldots, k$. This can be achieved by first transforming $\mathcal{C}^*$ into disjunctive normal form and then by applying to each of its disjuncts rules of the following types, until each of them contains only normalized **MLSS**-literals:

– if $i_0, i_1 \in \alpha \subseteq \mathcal{P}(\{1, \ldots, n\})$, then replace a literal of the form $v_\alpha = \bigcup_{i\in\alpha} x_i \setminus \bigcap_{j\notin\alpha} x_j$ by the set of literals

$$\{v_\alpha = z_1 \setminus z_2,\, z_1 = \bigcup_{i\in\alpha} x_i,\, z_2 = \bigcap_{j\notin\alpha} x_j\},$$

where z_1 and z_2 are newly introduced variables;

– replace each literal of the form $x = y \cap z$ by the set of literals

$$\{x = y \setminus w,\, w = y \setminus z\},$$

where w is a newly introduced variable;

- replace each literal of the form $x \subseteq y$ by the literal $y = y \cup x$;
- etc.

In addition, by a careful analysis of the translation algorithm in Figure 1, it is easy to see that

- all variables of the form v_α, where $\alpha \in \mathcal{P}^+(\{1, \dots, n\})$, must be in ψ_i',
- if $\mathcal{B}$ is a model of ψ_i' and U is the set of variables $\{v_\alpha : \alpha \in \mathcal{P}(\{1, \dots, n\})\}$, then ψ_i' satisfies the conditions (a) and (b) of Lemma 33,

for $i = 1, \dots, k$.

Since we are assuming that $\mathcal{C}^*$ is satisfiable, at least one of the ψ_j''s must be satisfiable, for some $j \in \{1, \dots, k\}$. Therefore, by applying Lemma 33 to ψ_j' (with respect to the set of variables $U = \{v_\alpha : \alpha \in \mathcal{P}^+(\{1, \dots, n\})\}$), it follows that there exists a model $\mathcal{M}$ of ψ_j' (and hence of $\mathcal{C}^*$) such that $|v_\alpha| \leq 1$, for each $\alpha \in \mathcal{P}^+(\{1, \dots, n\})$, thus proving the lemma. $\square$

6. Conclusions.

We presented a decision procedure for the set-theoretic sublanguage of set theory **MLSSmf** extending **MLSS** with constructs for expressing monotonicity, additivity, and multiplicativity properties of set-to-set functions. The decision procedure consists of a reduction algorithm which maps each sentence of **MLSSmf** into an equisatisfiable sentence of **MLSS**. Then the decidability of **MLSSmf** follows from the decidability of **MLSS**.

Our work can have applications in an interactive proof environment in which the user helps the system by telling which expressions are monotonic, while our decision procedure performs the tedious combinatorical steps. For instance, when proving the validity of the formula

$$\{f(x) : x \in a \setminus v\} \subseteq \{f(x) : x \in (a \cup b) \setminus v\}, \tag{7}$$

the user can instruct the system with the insight that the function

$$F(u) = \{f(x) : x \in u \setminus v\}$$

is increasing in u. Then, the system would conclude that to prove that is valid, it suffices to prove that

$$inc(F) \rightarrow F(a) \subseteq F(a \cup b) \tag{8}$$

is valid. Since (8) is an **MLSSmf**-formula, its validity can be automatically proven by our decision procedure.

Future directions of research may involve extensions of our decision procedure to handle other constructs related to set-to-set functions, such as injectivity and surjectivity of functions, as well as a fixed-point operator on monotone functions. Moreover, we are currently working on singling out convenient syntactic restrictions which allow a speed-up of the reduction process.

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